DCS/CSCI 2350: Social & Economic Networks

How are networks formed in the real world?

Modeling Networks

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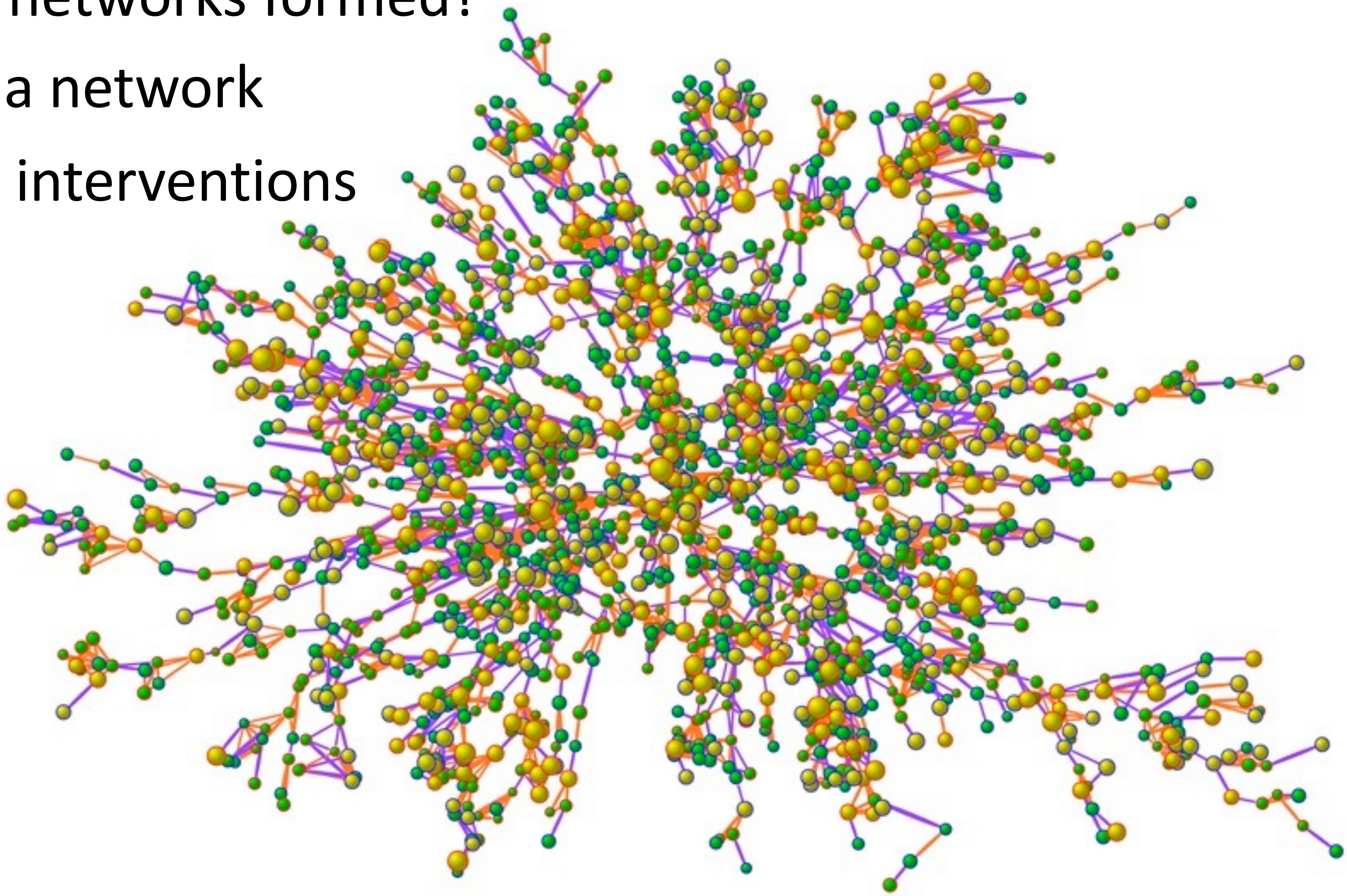
Reading

- Newman's *Networks,* Ch 12 (Canvas)
  - Erdos-Renyi random graphs
- Selected topics: Chapters 1, 4, 5 of Jackson's Social and Economic Networks book (Canvas)
  - Watts-Strogatz and preferential attachment
- Optional: Chapters 3, 4 of Watts's Six Degrees book (for behind the scene)

Why model networks?

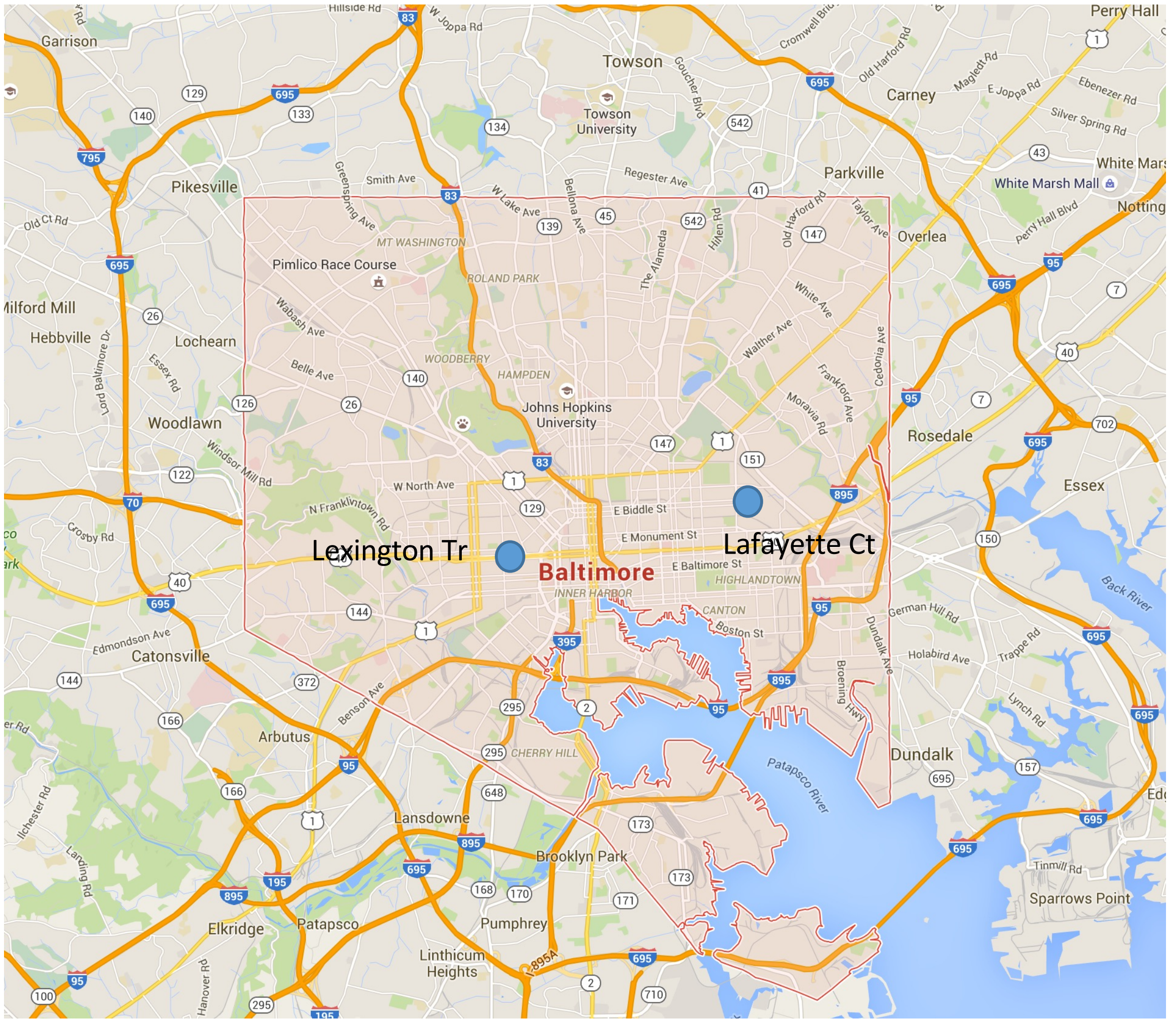
- How are networks formed?
- Effect of a network
- Targeted interventions



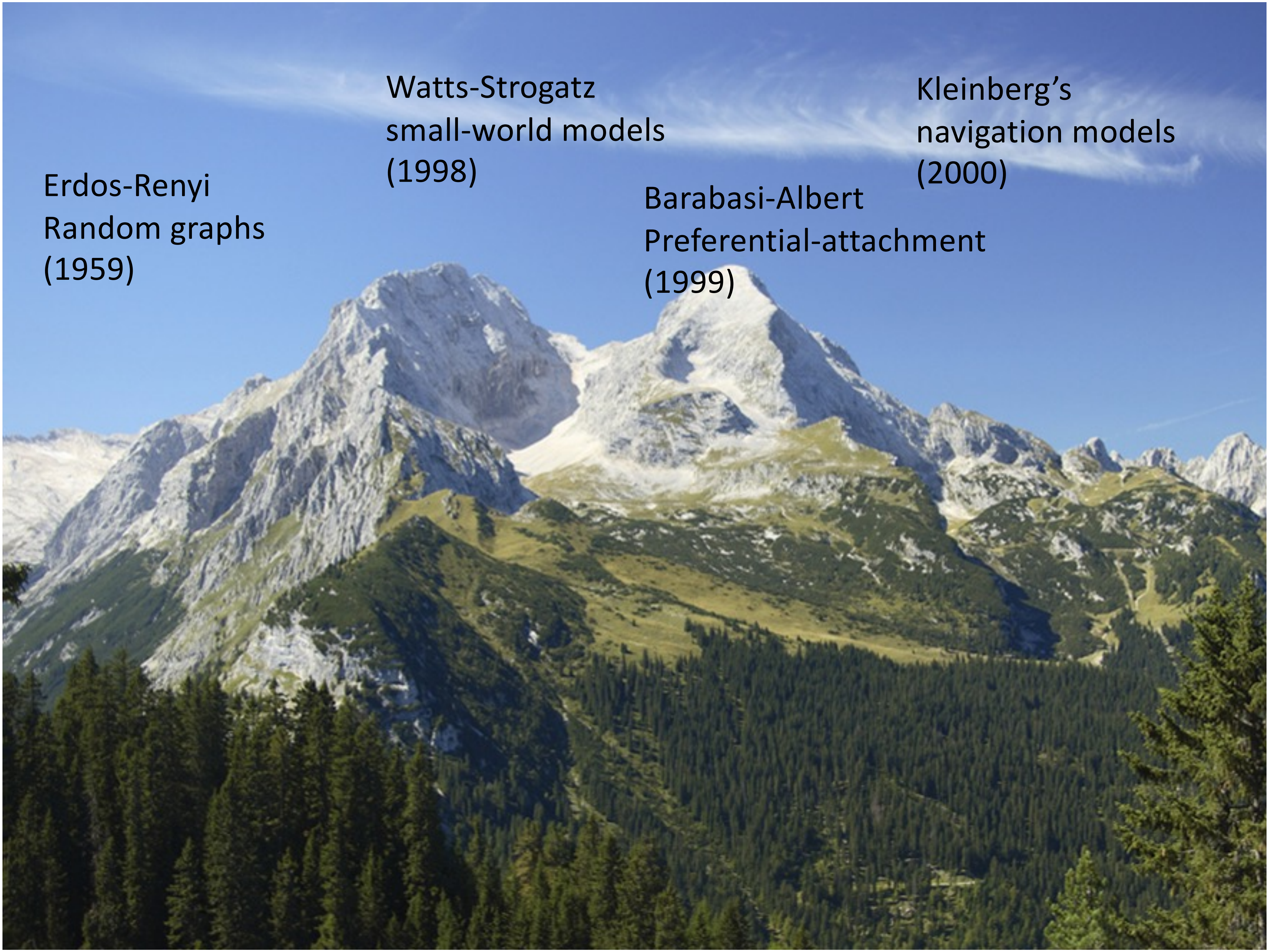
Hush puppies (1995)

Baltimore STD outbreak (1995)

1995 to 1996: 500% increase



Iconic photo of Lafayette Court housing


Watts-Strogatz
small-world models
(1998)
Kleinberg's
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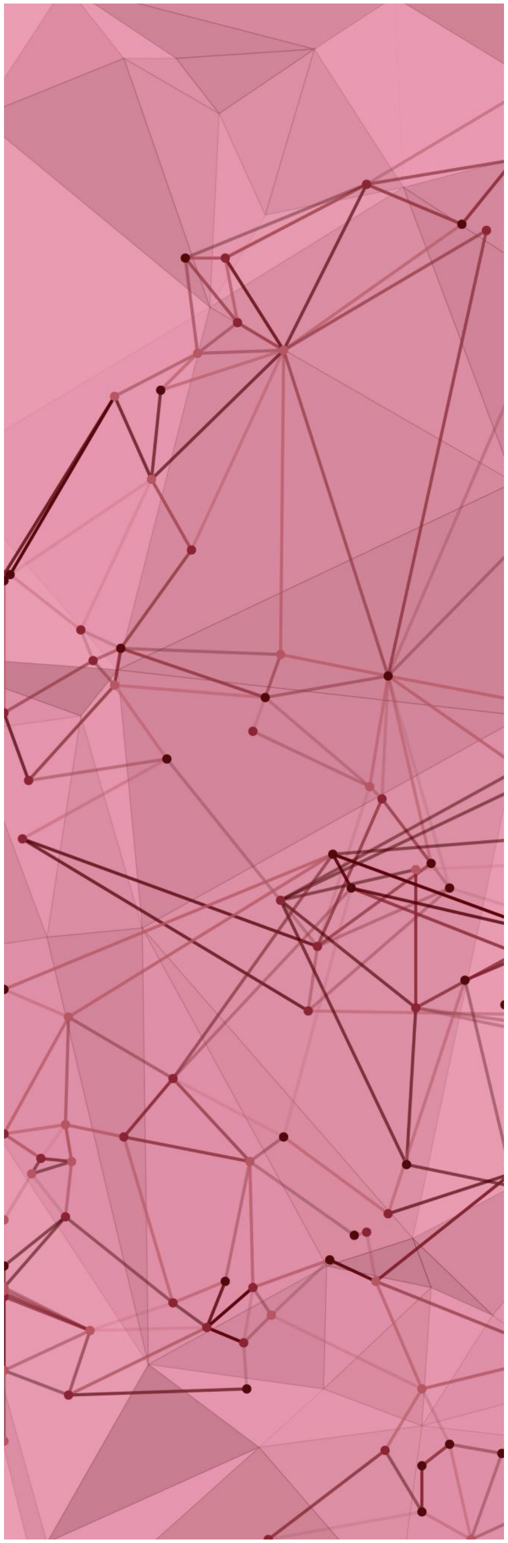


Why random network models?

Why randomization?

- First attempt -> richer models
- Powerful!
  - An ensemble of networks, not just one network
  - Can capture real-world network properties
- Context-free
- Computationally tractable

Prelude

- Power-law degree distribution, $p_k = C\, k^{-\alpha}$

- (Puzzle 2)
  Max possible # of edges = $\binom{n}{2} \equiv \frac{n(n-1)}{2}$

- Formula: $\binom{n}{k} \equiv \frac{n!}{k!\,(n-k)!}$

Erdos-Renyi random graphs (or random graphs)

- Static
  - Given n nodes (constant)
- **Model 1**
  - Inputs: number of nodes n and probability of forming an edge = p
  - Each pair of nodes is connected by an edge with prob. p
- Model 2
  - Inputs: number of nodes n and number of edges m
  - Create m edges uniformly at random out of $\binom{n}{2}$ total possible edges

Properties of Erdos-Renyi graphs

- Every simple graph is possible!
- How can we say something regarding properties?
  1. Estimate the probability of a property
  2. Limiting behavior: n → infinity

Properties of Erdos-Renyi graphs

- Degree distribution
- Clustering coefficient
- Small-world effect
- Giant component

Degree distribution

- $p_k = e^{-c}\, c^k / k!$ [Poisson distribution]
- Here, mean degree $c = p(n-1)$
  [AKA average deg. or expected deg.]

- Power-law degree distribution,
  $p_k = C\, k^{-\alpha}$
- (Puzzle 2)
  Max possible # of edges =
  $\binom{n}{2} \equiv \frac{n(n-1)}{2}$
- Formula: $\binom{n}{k} \equiv \frac{n!}{k!\,(n-k)!}$

Derivation (optional)

$$P_k = \binom{n-1}{k} p^k (1-p)^{n-1-k}$$

$$= \frac{(n-1)!}{(n-1-k)!\,k!}\, p^k (1-p)^{n-1-k}$$

$$= \frac{(n-1)(n-2)\cdots(n-k)\,\cancel{(n-k-1)!}}{\cancel{(n-1-k)!}\;k!}\, p^k (1-p)^{n-1-k}$$

$$\simeq \frac{(n-1)^k p^k}{k!} (1-p)^{n-1-k}$$

$$= \frac{c^k}{k!} (1-p)^{n-1-k}$$

$$\simeq \frac{c^k e^{-c}}{k!}.$$

$$\ln (1-p)^{n-1-k} = (n-1-k)\ln(1-p)$$

$$\simeq -p(n-1-k)$$

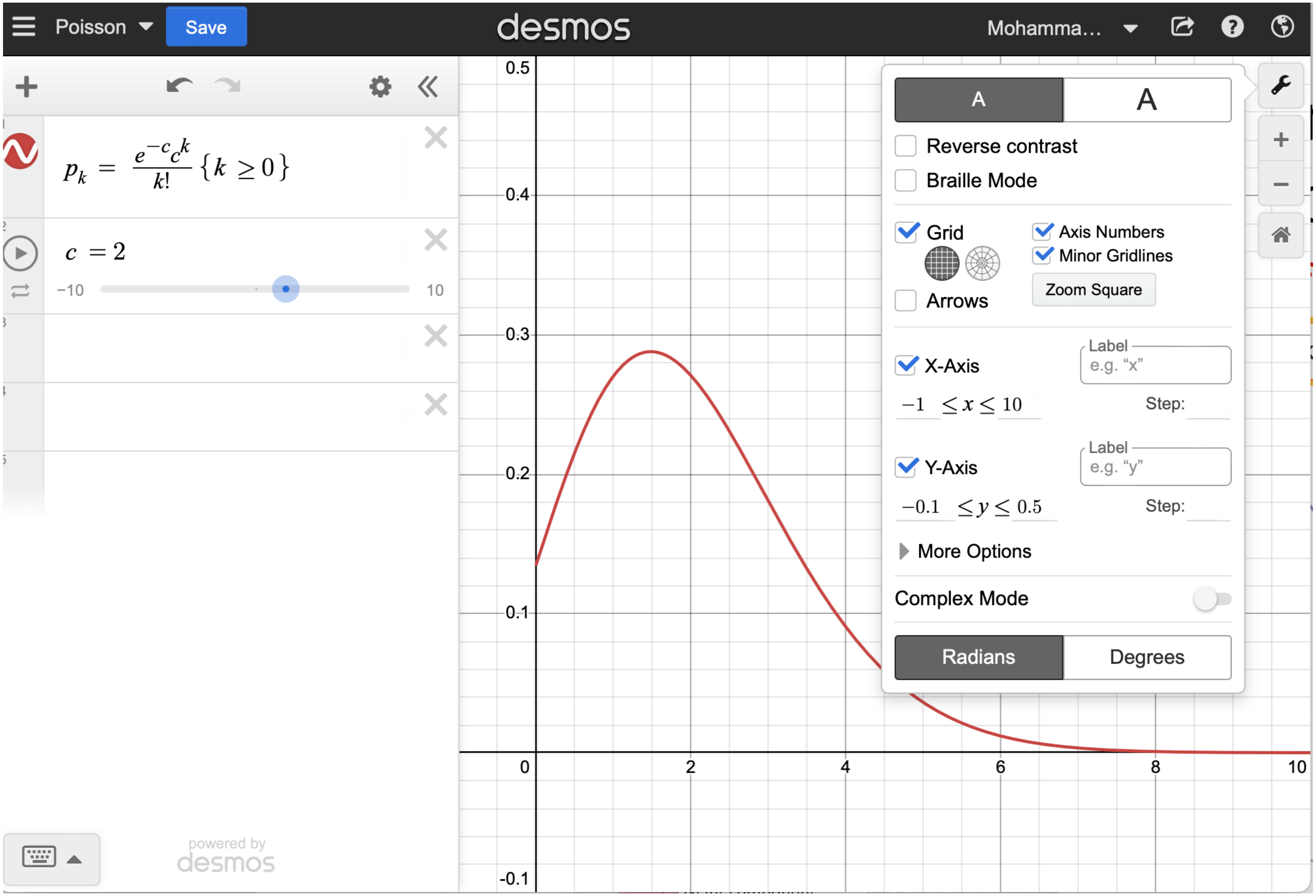
$$\simeq -p(n-1)$$

$$= -c$$

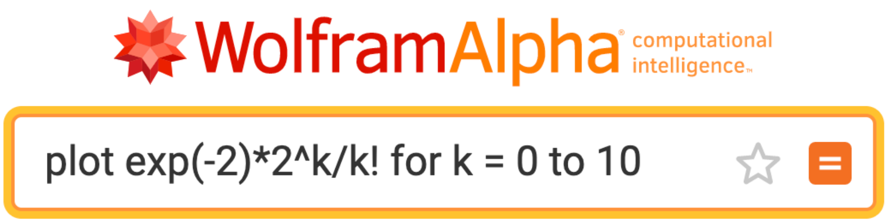
$$\Rightarrow (1-p)^{n-1-k} \simeq e^{-c}$$

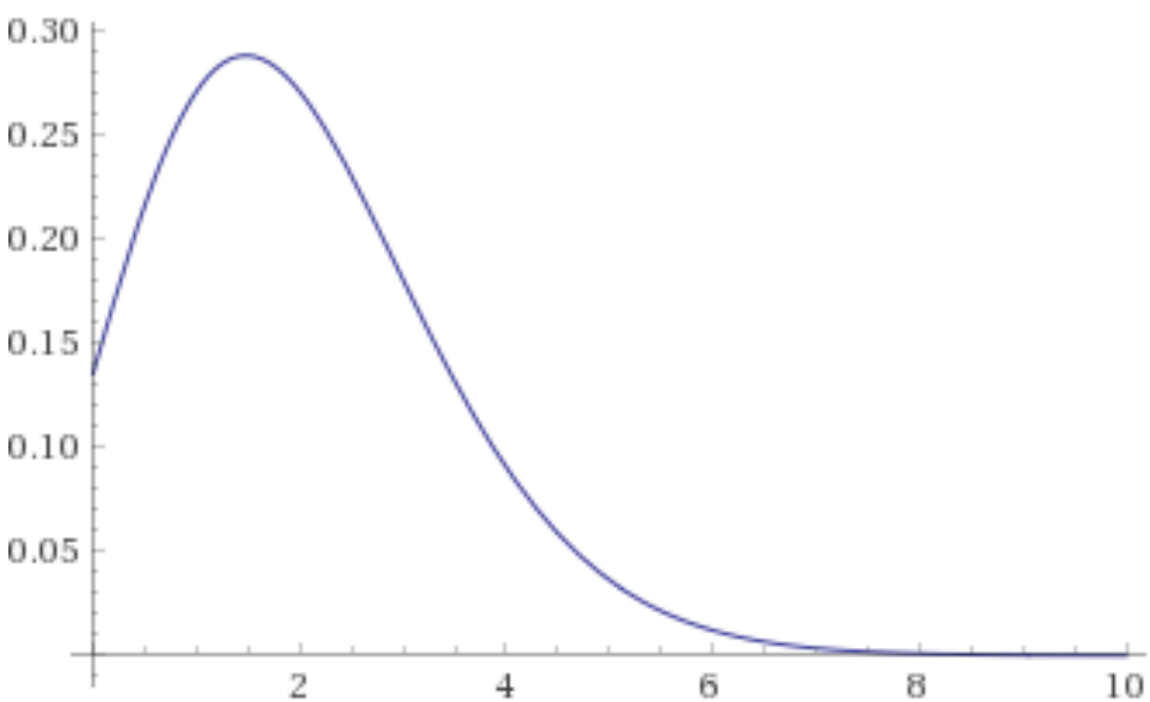
Approximate using the Maclaurin series: ln (1-p) = – p – p2 – p3 –… and discarding the latter terms to get ln(1-p) ≃ –p

Plotting Erdos-Renyi degree distribution



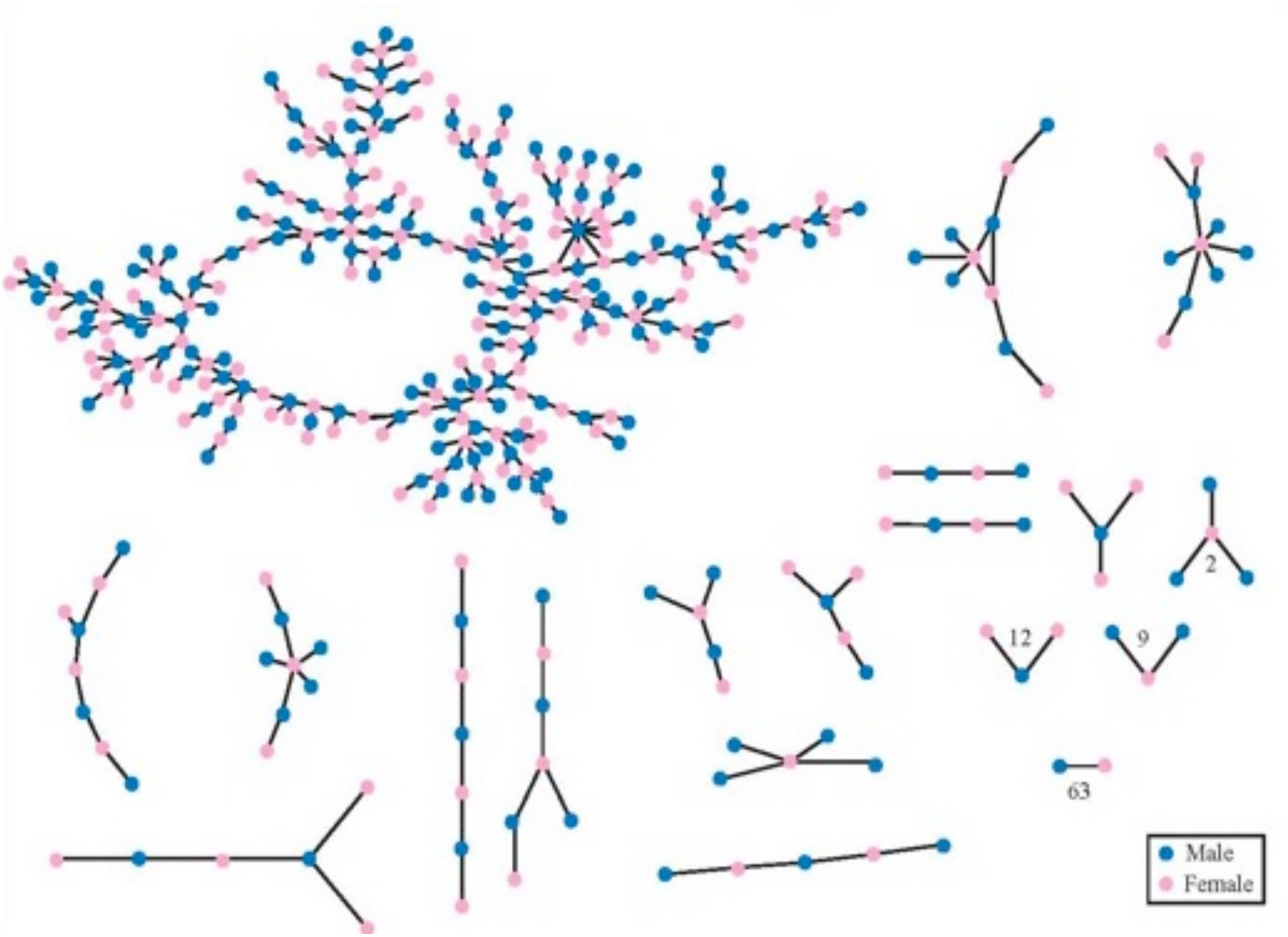
Plotting Erdos-Renyi degree distribution



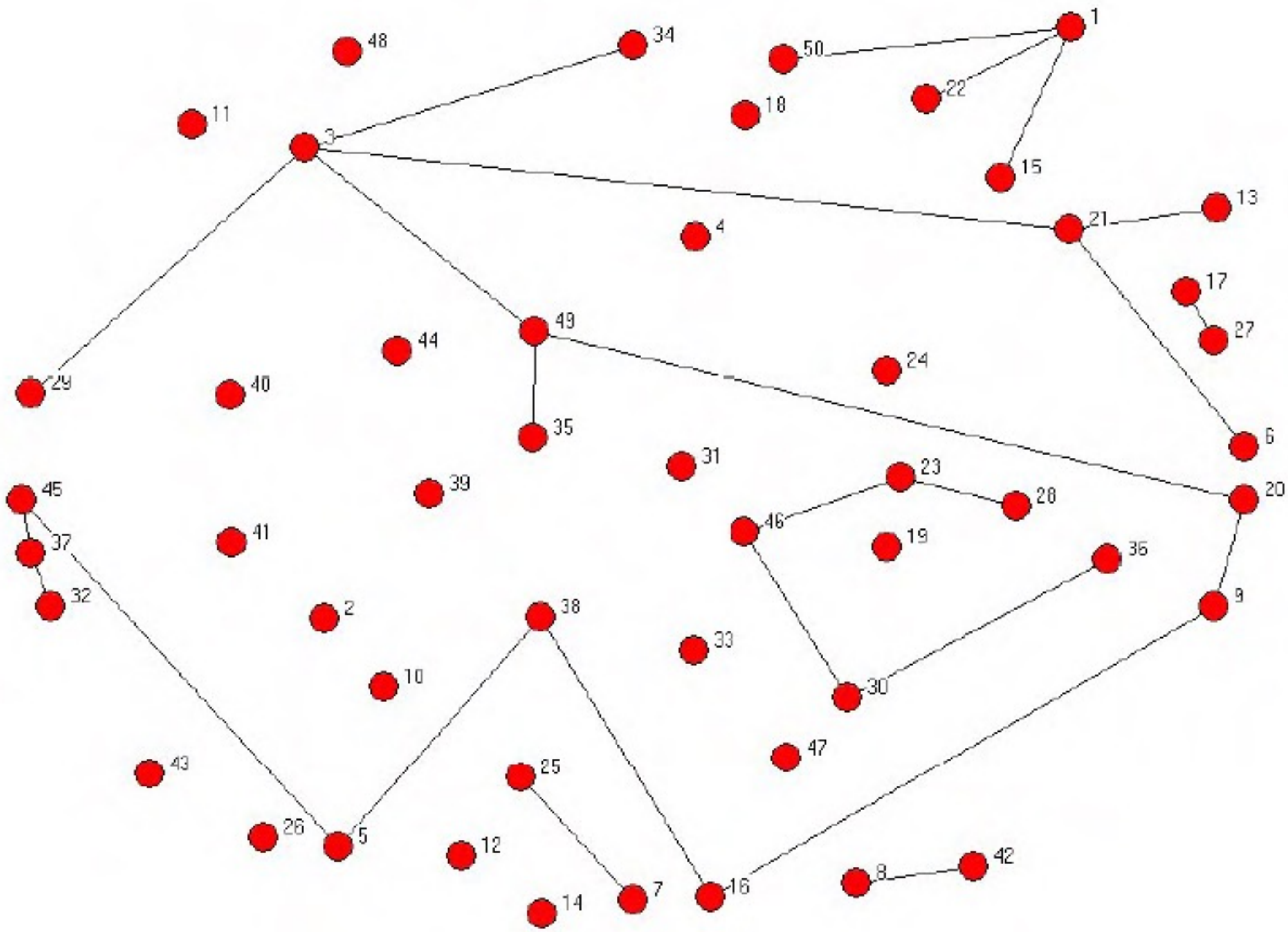


- Plug in Poisson distribution
- Expected degree, c = 2

High-school relationships (Bearman et al, 2004)

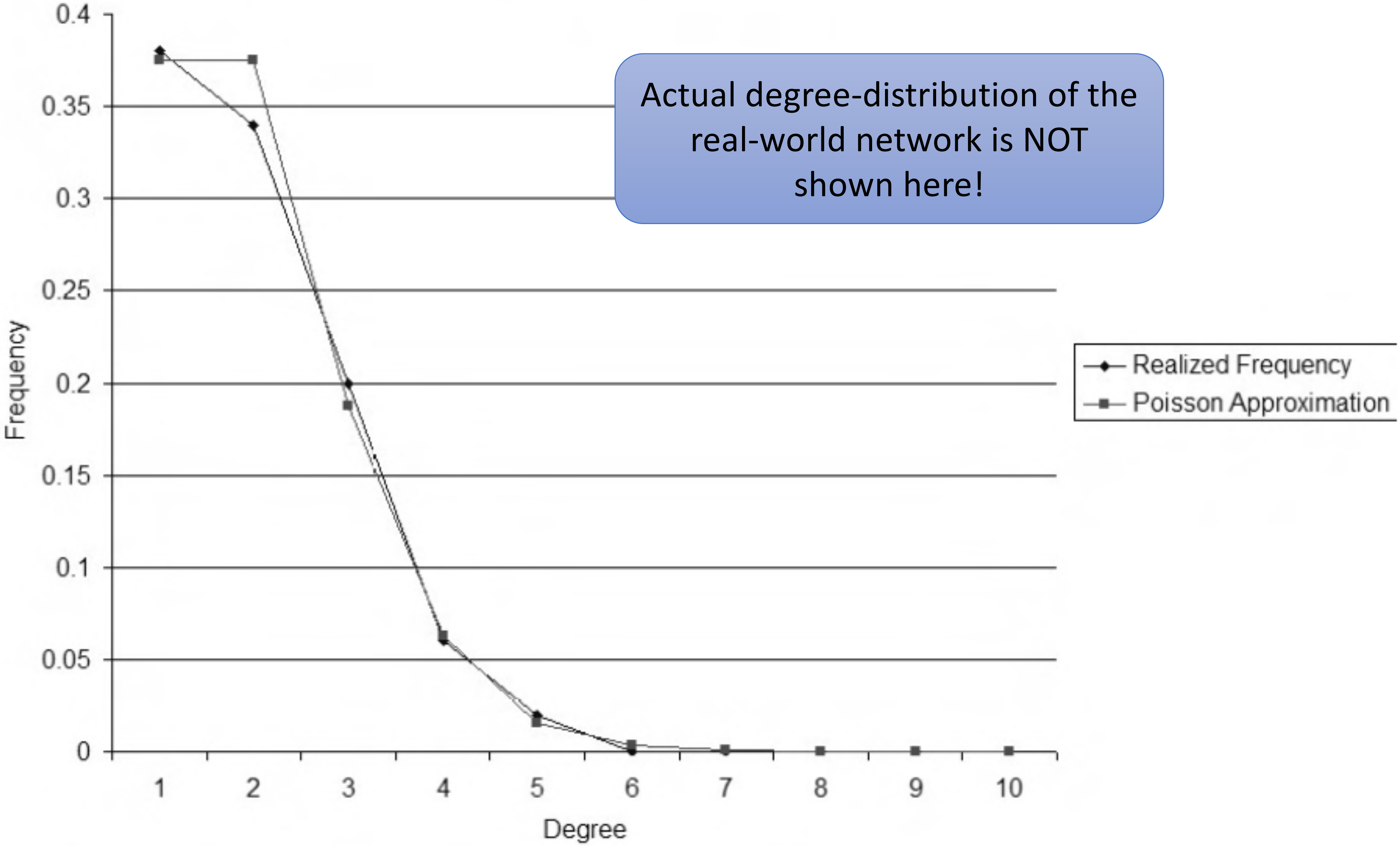


Random graph with p = 0.02



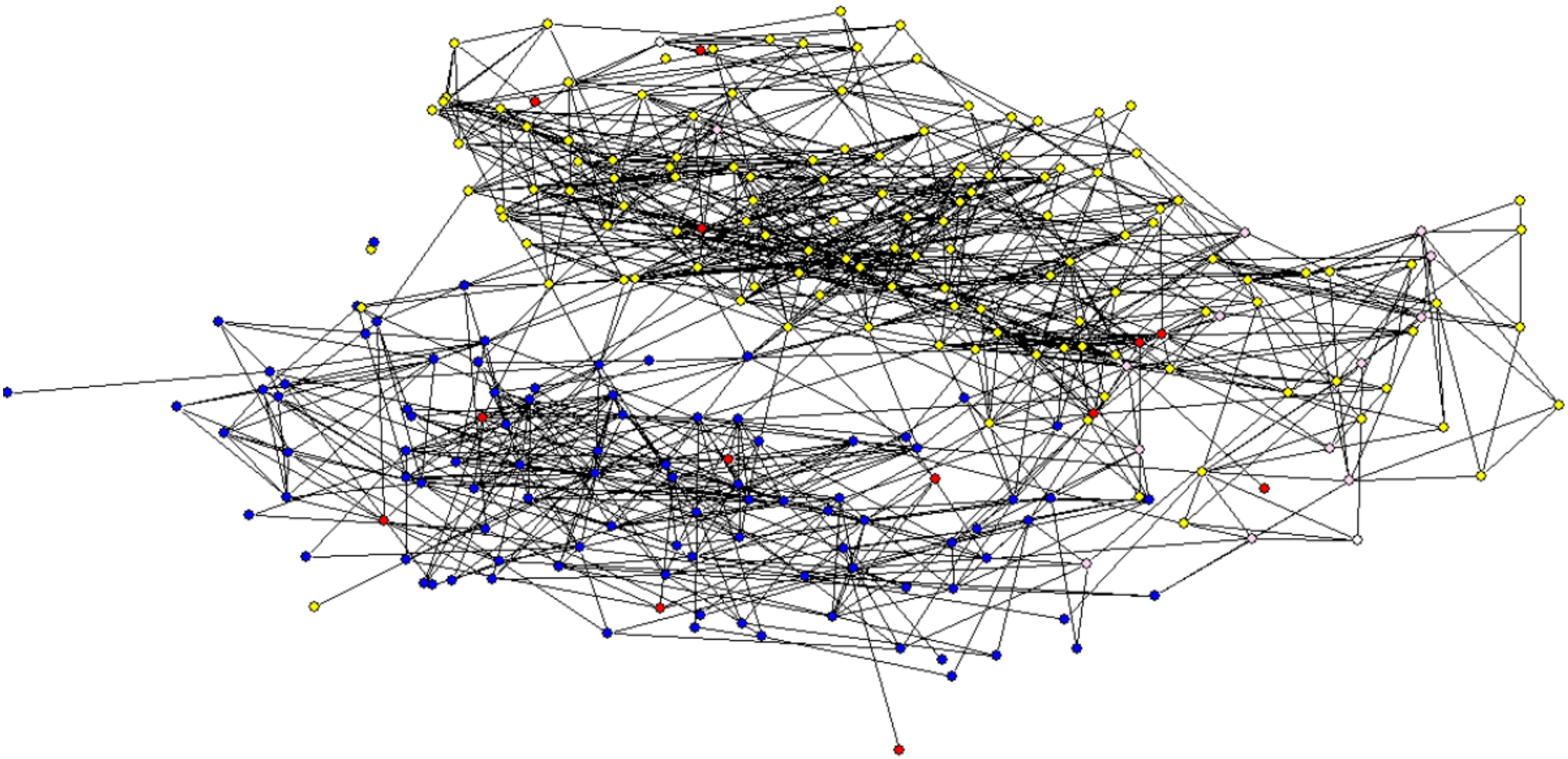
Degree distribution: p = 0.02

Approximation by Poisson distribution

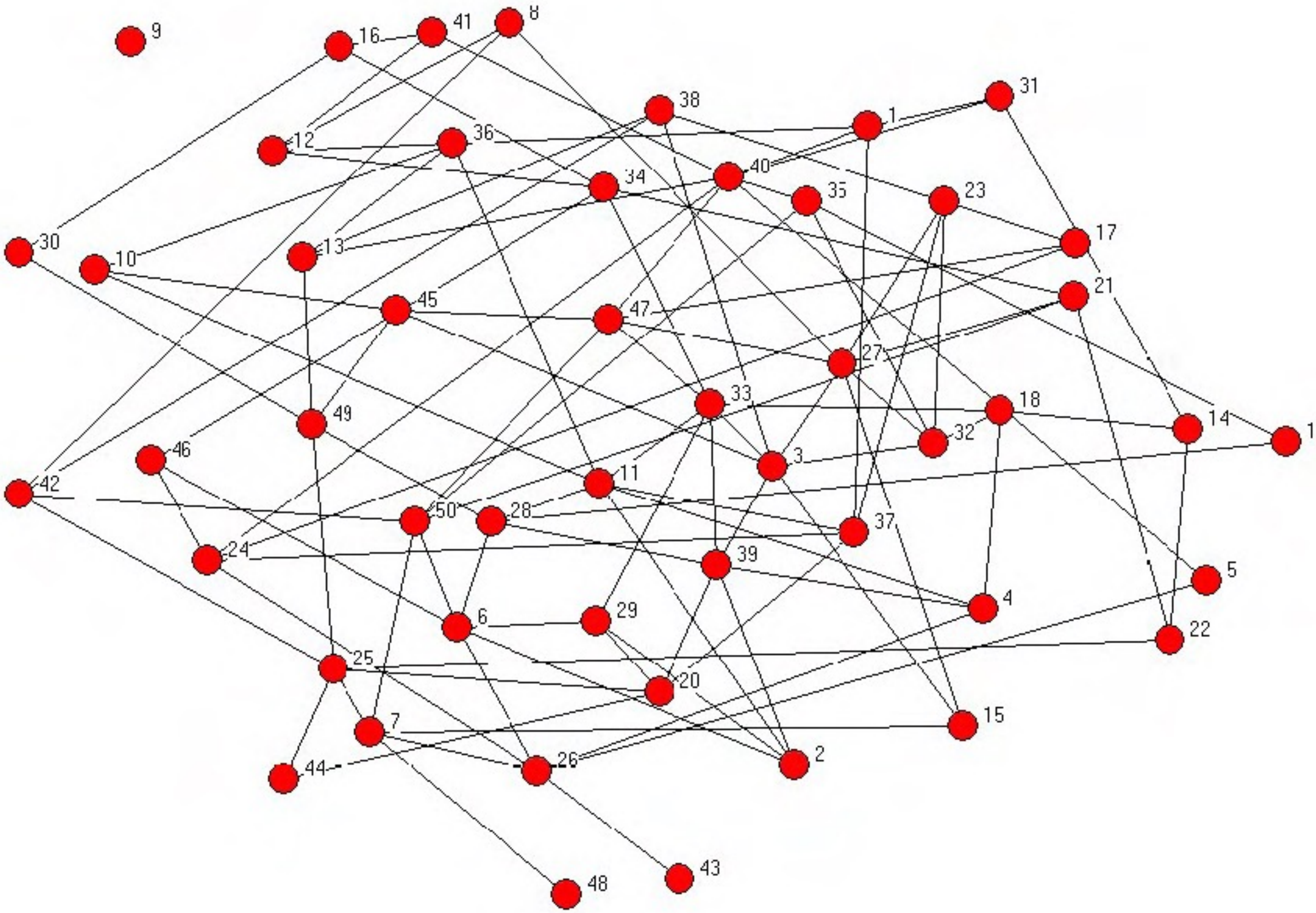


Actual degree-distribution of the real-world network is NOT shown here!

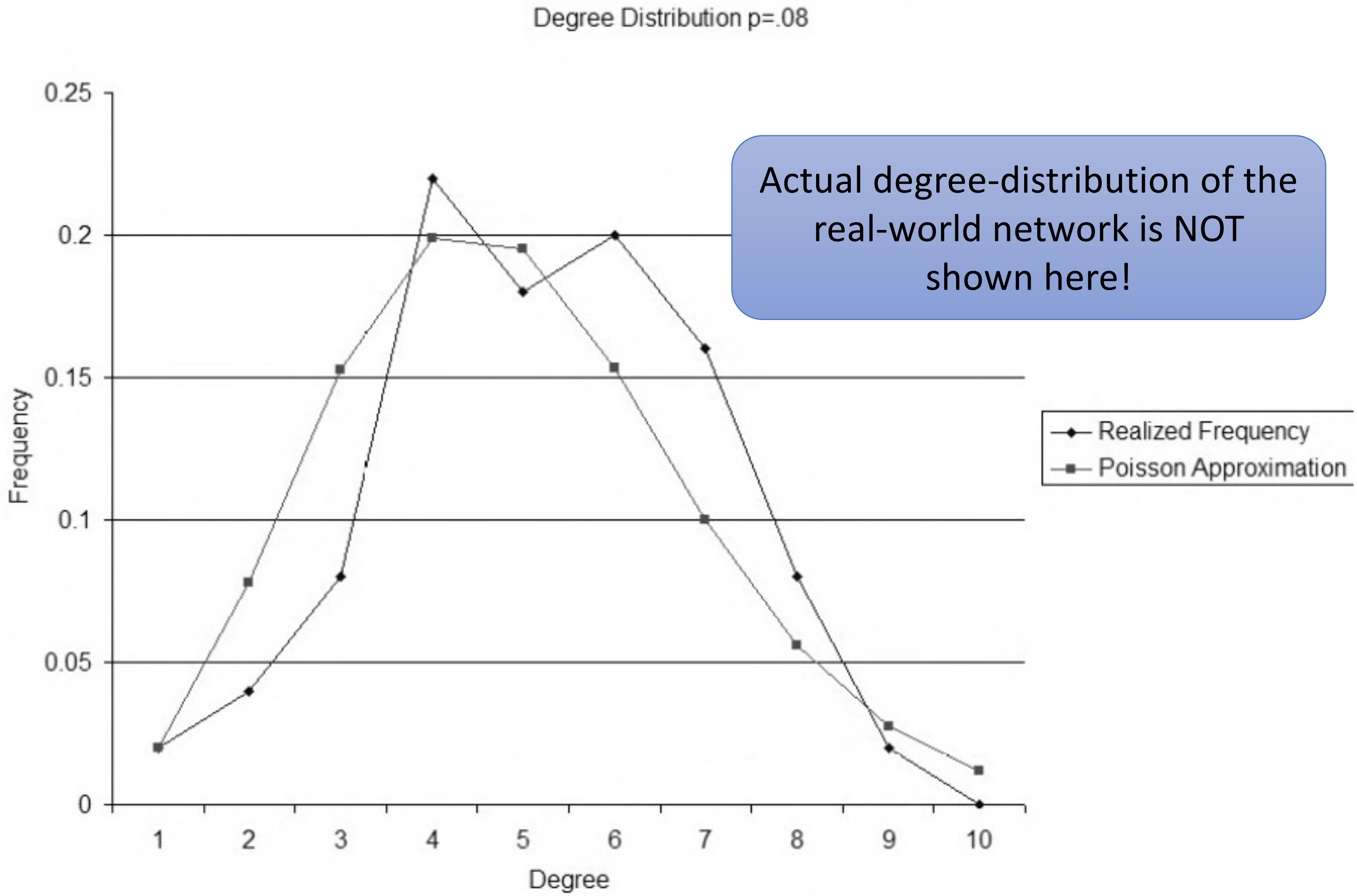
High-school friendships (Currarini et al, 2007)



Random graph with p = 0.08



Degree distribution: p = 0.08



Actual degree-distribution of the real-world network is NOT shown here!

Erdos-Renyi: Giant component

Phase transition & giant comp (GC)

Let q fraction of nodes be in the GC:

Fraction of nodes outside of the GC = 1-q

Prob of finding a node outside of the GC irrespective of its degree = right hand side below

$$1-q = \sum_k p_k (1-q)^k$$

$$1-q = \sum_k \frac{e^{-c} c^k}{k!} (1-q)^k$$

$$= e^{-c} \sum_k \frac{[c(1-q)]^k}{k!}$$

$$= e^{-c} e^{c(1-q)}$$

$$= e^{-cq}$$

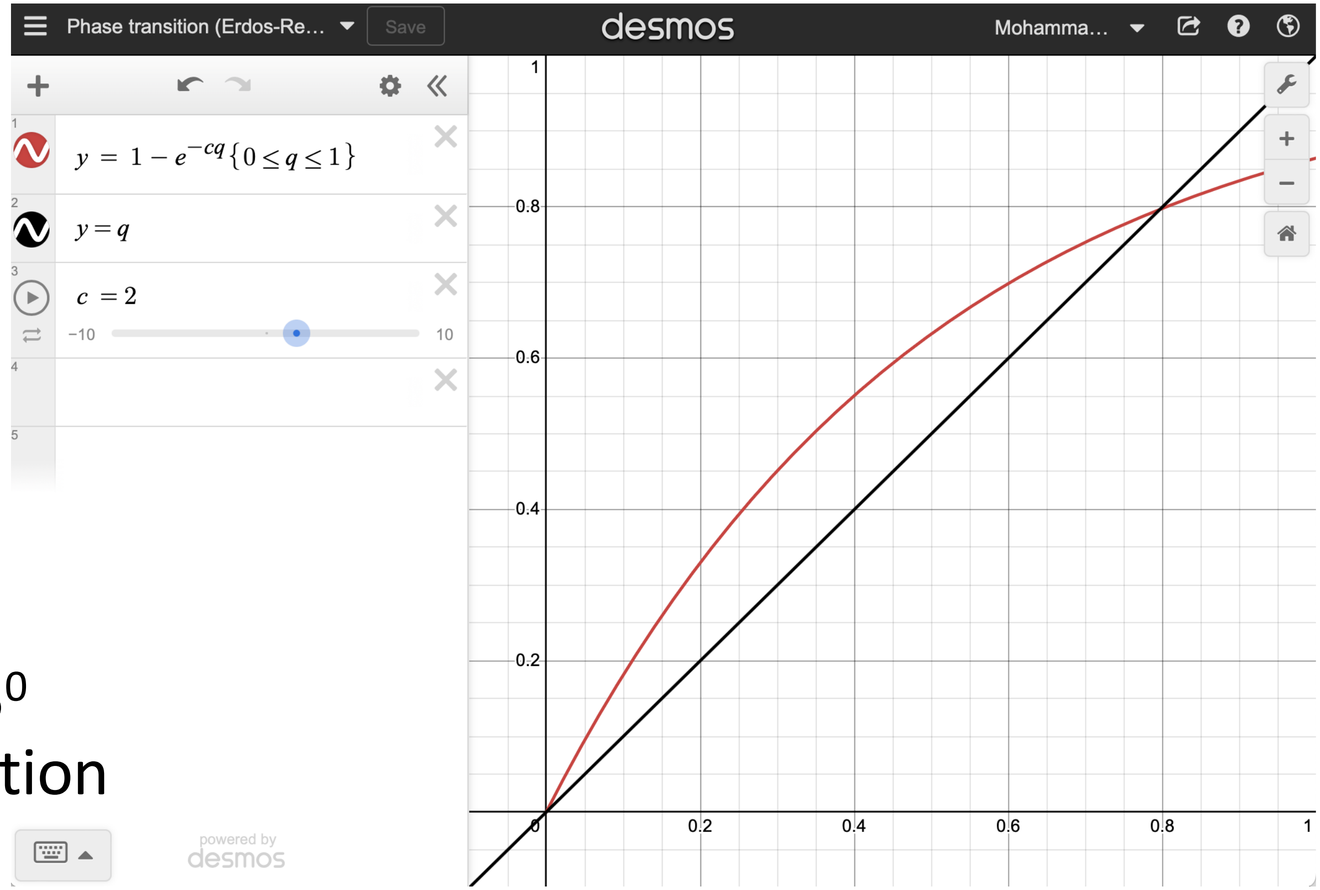
$$1-q = e^{-cq}$$

$$\boxed{q = 1-e^{-cq}}$$

Experimental solution:

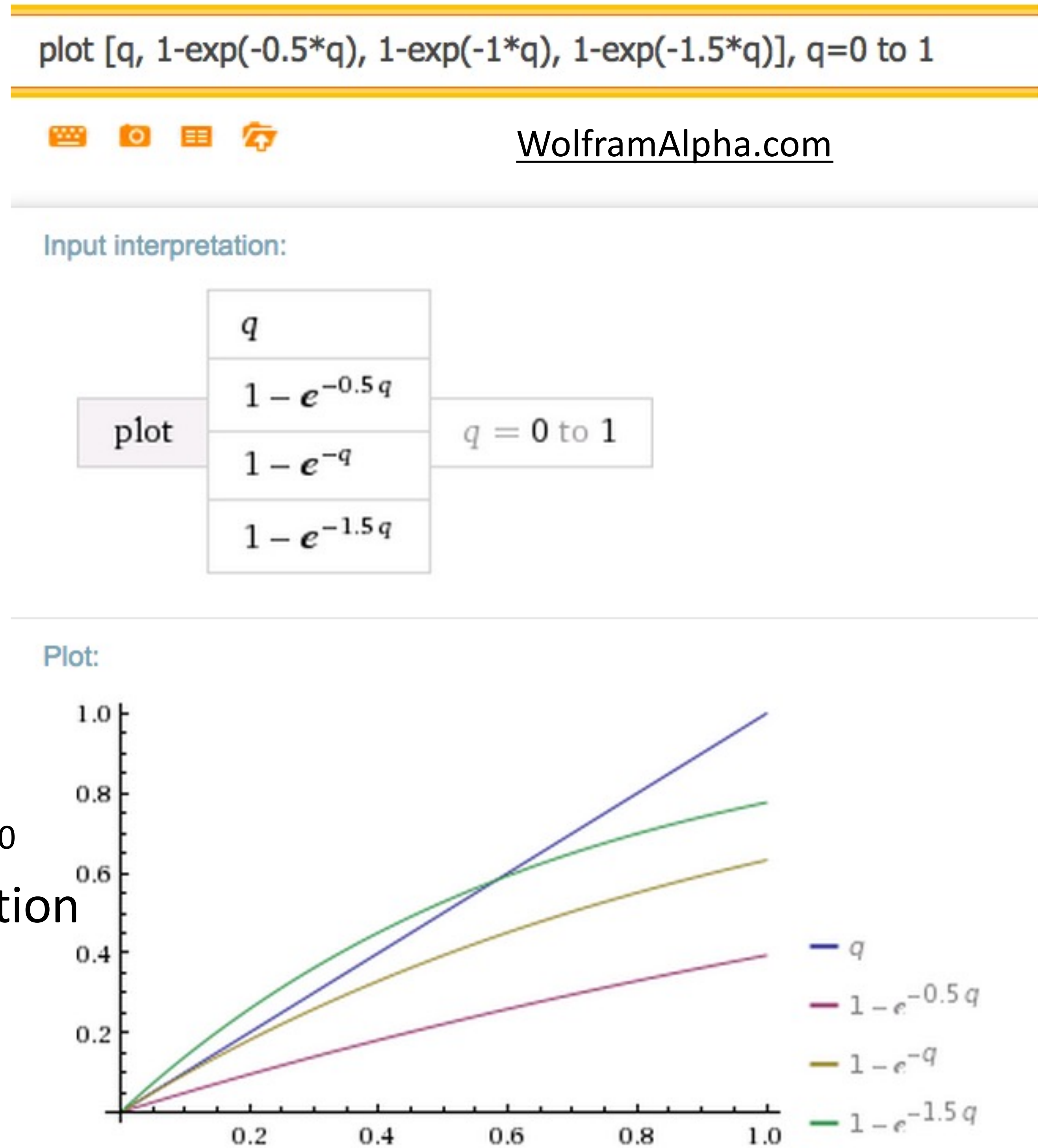
<u>GC emerges when c > 1</u>

Phase transition & giant comp.



- $q = 1 - e^{-cq}$
  - X-axis is q
  - Y-axis is $1 - e^{-cq}$
  - Intersection with 45^0 line solves the equation
- Giant component emerges when $c > 1$

Phase transition & giant comp.



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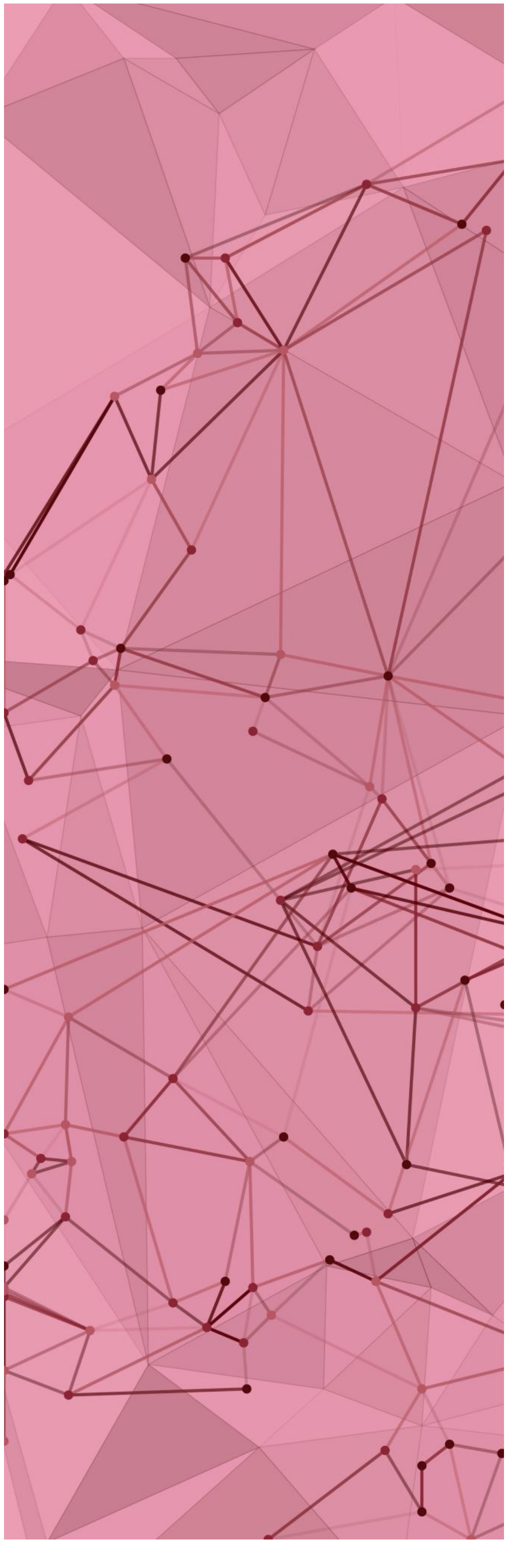
Giant component: Netlogo experiments

1. Prof. Irfan's program: https://mtirfan.com/Erdos-Renyi.html

2. Netlogo -> Models Library -> Networks -> Giant Component

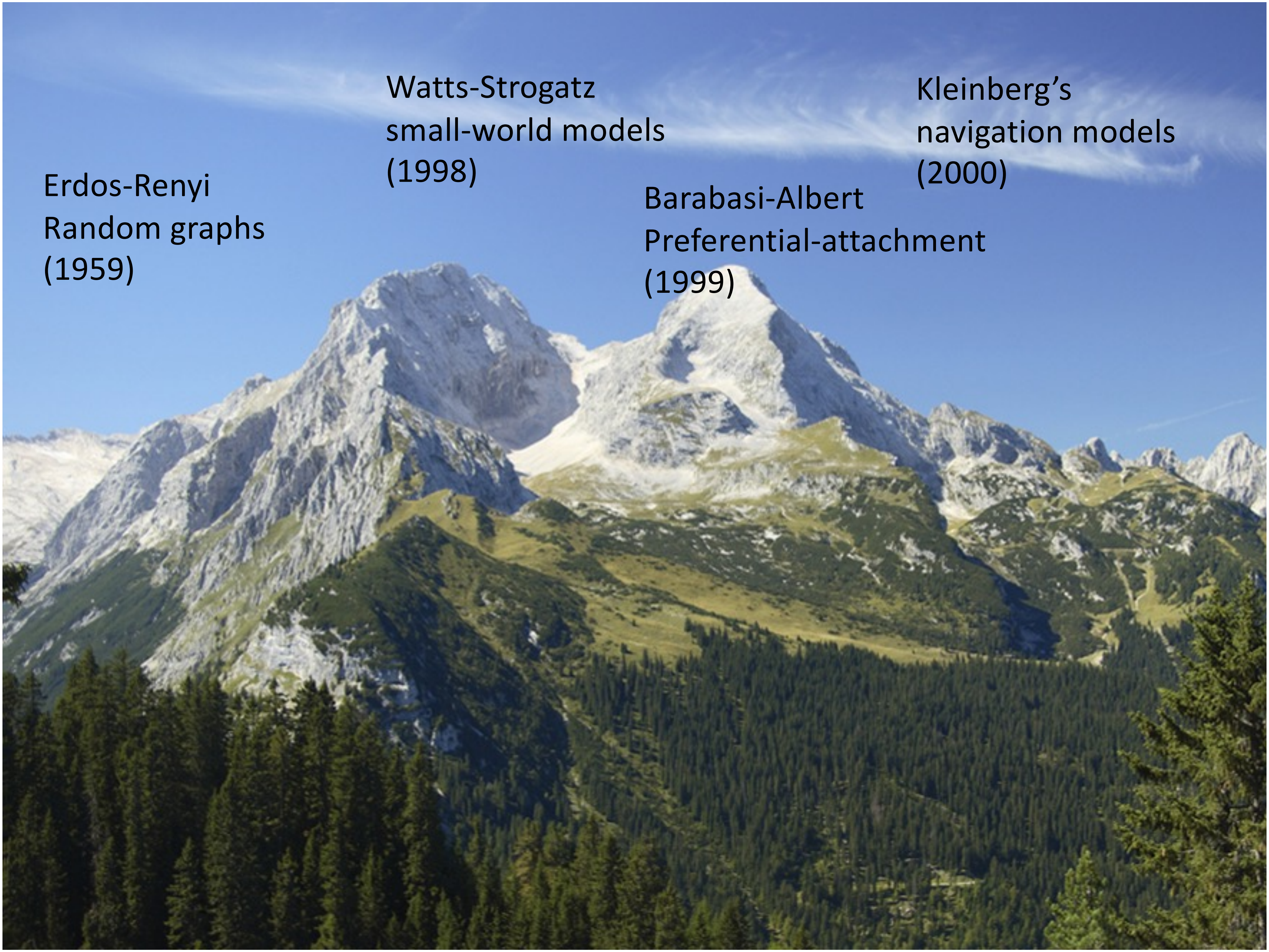


Erdos-Renyi: Clustering coefficient

Erdos-Renyi:
Small-world property

Properties of Erdos-Renyi graphs

- Degree distribution 
- Giant component 
- Clustering coefficient 
- Small-world effect 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Watts-Strogatz small worlds model

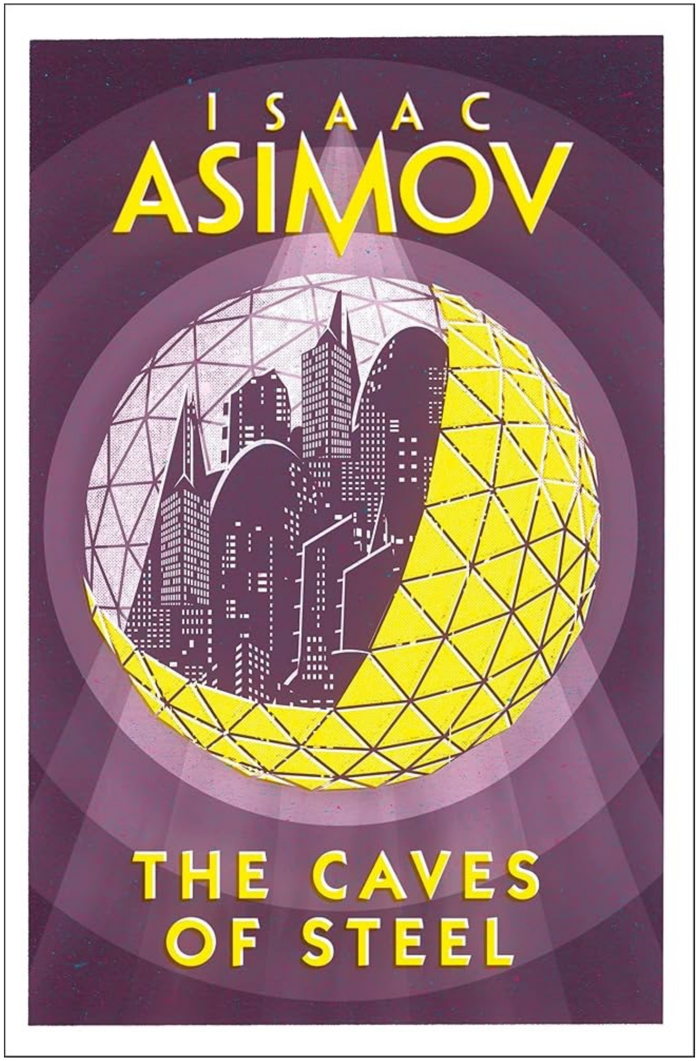
Question

How to create random graphs that capture the real-world clustering properties?

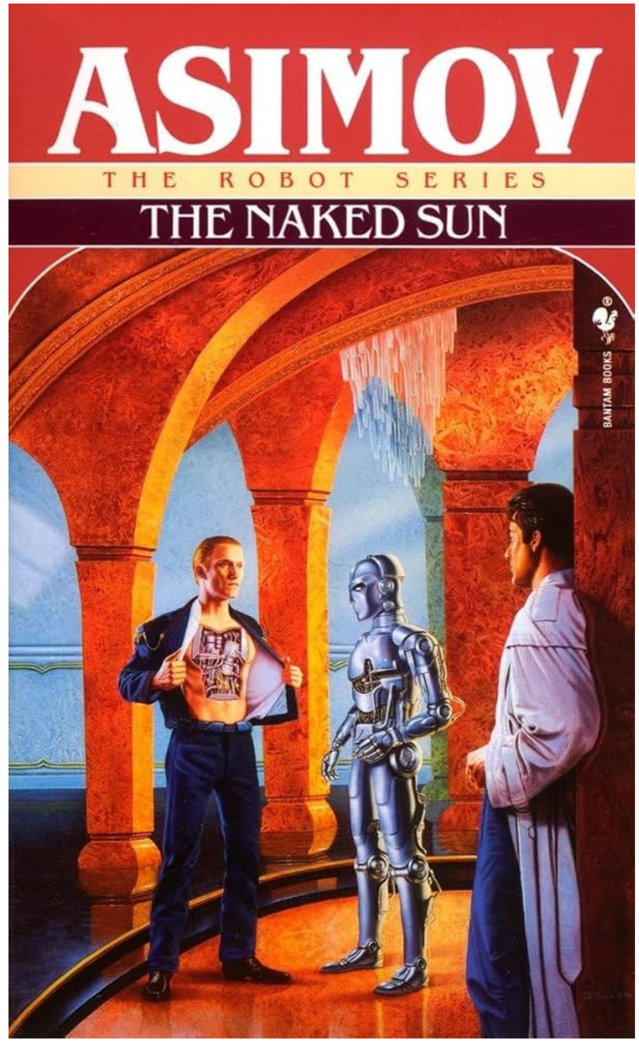
Watts-Strogatz small worlds model

Inspiration: Isaac Asimov

Edges within each cave



Solaria: random edges



For this simple model, one surprising result is that on average, the first five random rewirings reduce the average path length of the network by one-half, *regardless of the size of the network*.

Duncan Watts, *Six Degrees*, pg. 89

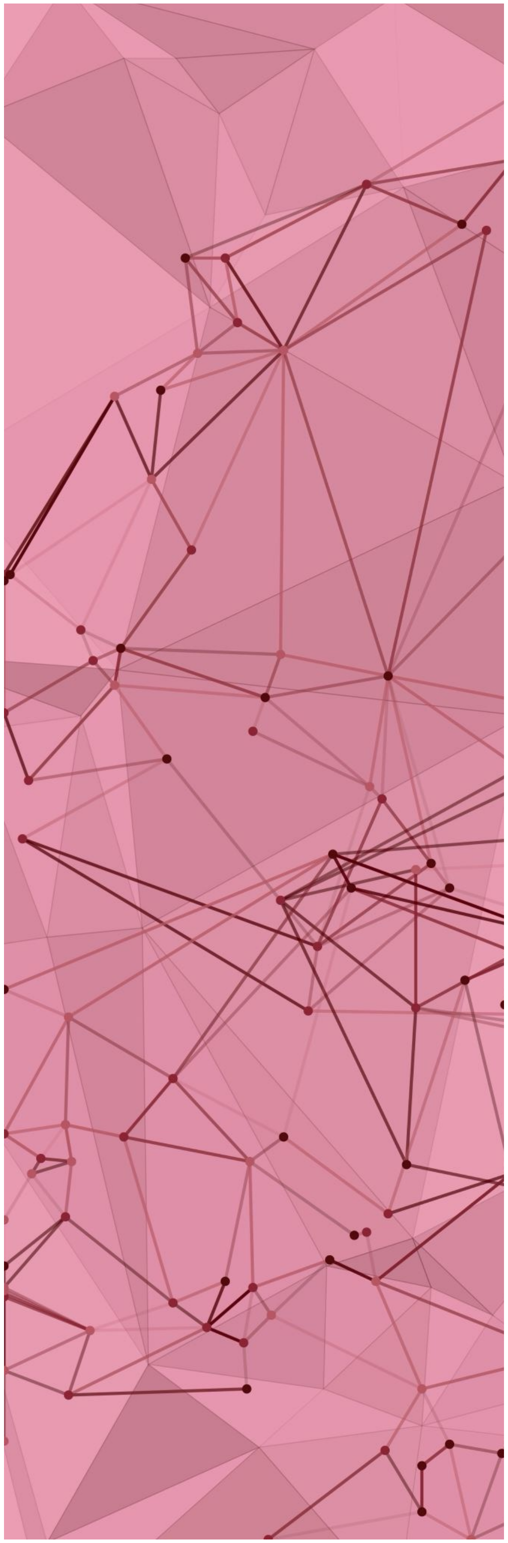
For this simple model, one surprising result is that on average, the first five random rewirings reduce the average path length of the network by one-half, *regardless of the size of the network*. The bigger the network, the greater the effect of each individual random link so the impact of adding links becomes effectively independent of size. The law of diminishing returns, however, is just as striking. A further 50 percent reduction (so that now the average path length is at one-fourth of its original value) requires roughly another fifty links—roughly ten times as many as for the first reduction and for only half as much over-

Duncan Watts, *Six Degrees*, pg. 89

Watts-Strogatz small-world model (1998)

- Degree distribution 
- Clustering coefficient 
- Giant component 
- Small-world effect 

Experiment with NetLogo:
File → Models Library →
Networks → Small Worlds



Barabasi-Albert preferential attachment model

Question

How to create random graphs that capture the real-world degree-distribution?

Barabasi-Albert preferential attachment model

Examples

- Pareto (1890s)
  - Wealth distribution, city sizes
- Herbert Simon (1955):
  - System grows over time with new objects entering
  - Existing objects grow proportional to their size
  - “The rich gets richer faster than the poor”
- Derek Price (1965)
  - Citation network: # of citations of a paper is proportional to the # of citations it has

Barabasi-Albert Preferential-attachment model (1999)

- Nodes are born over time (only one node at a time). DOB: {0, 1, 2, ..., t, ...}
- Degree of node i at time t: $d_i(t)$
- Upon birth, a node forms M edges with existing nodes with prob proportional to the existing nodes' degrees

M is the only model parameter!

Preferential attachment

Degree distribution is power law!

(derivation)

Barabasi-Albert Preferential-attachment model (1999)

- Degree distribution 
- Clustering coefficient ?
- Giant component 
- Small-world effect 

Experiment with NetLogo:
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